Economic Capital Modeling
Closed form approximation for real-time (pricing) applications

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The views and opinions expressed in this presentation are those of the authors and do not necessarily reflect those of the European Investment Bank or the European Investment Bank Institute. All figures shown are based on purely hypothetical test portfolios.
Overview

1. A Primer on Economic Capital
2. Regulatory vs. Economic Capital
3. Model framework
4. Closed form approximation
5. Performance on a homogeneous test portfolio
6. Performance on a heterogeneous test portfolio
7. Conclusions
1. A Primer on Economic Capital

- Economic Capital (ECap) is a *risk metric* designed to measure the riskiness of an entity’s (called ”Bank” in the sequel) business activities and quantify the level of capital commensurate to such risk.
- As such, it is the output of a *mathematical model*.
- Here, we restrict ourselves to the *credit risk* a Bank accepts as a result of its (contingent) investment activity: credit risk on loans, treasury holdings, counterparty credit risk from derivative activities etc.
- Market risk and operational risk\(^1\), legal risk and reputational risks, while do contribute to a Bank’s overall ECap figure, are excluded in this presentation.

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Pillar 2 of the Basle Regulatory Framework requires Banks to perform a so-called *Internal Capital Adequacy Assessment Process (ICAAP)*.
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Given a desired confidence level, an example of a sufficiently capitalized Bank looks as follows:

![Example Balance Sheet](image-url)
1. A Primer on Economic Capital

An example of a (for a desired confidence level) not sufficiently capitalized Bank looks as follows:

![Example Balance Sheet]

- **Risky Assets**: 100
- **Liabilities (Debt)**: 80
- **Capital**: 20
- **Economic Capital (Ecap)**: 24

The balance sheet shows that the "Book Capital" = 20 EUR is insufficient to cover the liabilities and risky assets. The capital ratio is calculated as follows:

\[
\text{Capital Ratio} = \frac{\text{Capital}}{\text{Liabilities}} = \frac{20}{80} = 0.25
\]

This ratio indicates that the bank has a capital adequacy ratio of 25%, which is below the required minimum for stability and solvency. The graph highlights the need for additional capital to meet regulatory requirements.
2. Regulatory vs. Economic Capital

Evidently, there is more than one mathematical model for calculating capital requirements.

External view

- Regulatory Capital reflects the regulator's view: Pillar 1 rule-based calculation, i.e. "one-size-fits-all formula"
- Rating Agencies’ Capital Figures reflect their views: typically one methodology for a subset of banks

Internal view

- Economic Capital reflects the Bank’s internal view in Pillar 2: more freedom in choice of model which is deemed most appropriate for the specific institution
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## Regulatory Capital

The AIRB approach is based upon:

1. Value-at-Risk ("How much capital do we need to survive?")
2. 99.9% confidence level (reflecting an A rating)
3. Time horizon $\tau_h = 1$ year
4. ASRF model with *portfolio invariance* (no concentration risk and no diversification benefits)
5. Only few types of collateral eligible for risk mitigation
6. Default correlation modeling with a single factor driving defaults
7. Possibly still "substitution approach" for guar. exp.

## Economic Capital

ECap can e.g. be based upon:

1. Expected shortfall ("What if we do not survive?")
2. Confidence levels more akin to AAA rating
3. Longer time horizon ("How quickly can one counteract a capital inadequacy?")
4. Models based on actual portfolio to reflect concentration risk and diversification benefits
5. Collateral mod. based on internal assessment of economic relevance
6. Default correlation modeling using multi-factor models
7. "Double default approach" for guaranteed exposures
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Regulators impose ICAAP as they know well the (restrictive) assumptions underpinning the Pillar 1 regulatory model. Banks have to have an internal model in place and typically use ECap as a key risk indicator. Depending on the institution, ECap is applied for:

- Monitoring capital adequacy and portfolio concentrations
- Business planning
- Stress testing
- Ex-post measurement of "risk-adjusted returns" across groups of transactions, business units etc.
- Ex-ante pricing: ECap does address concentration and is, thus, key for unexpected loss pricing.
3. Model framework - general setting

- Estimates for default probabilities \( p_i \) and "loss-given-default" (LGD) parameters \( Q_i \) are assumed to be given. But how can we describe the simultaneous default behaviour of obligors?

- Merton-based models are inspired by the balance sheet view: obligor \( i \) defaults if its asset value \( \bar{X}_i = 100 \) drops below a default threshold \( \bar{d}_i = 80 \).

- The asset values \( \bar{X}_i \) of different obligors may be driven by some common market factors \( \bar{Y}_i \) and some un-systematic, obligor-specific components \( \varepsilon_i \). Dependence on common market factors introduces default correlation.

- All fine, but how do we model the asset values (returns) then?
  - Asset returns are not observable, which "proxies" to take for \( X_i \)?
  - How many factors drive the asset return \( X_i \) of obligor \( i \)?
  - Which factors \( Y_i \) are the "right ones" to take?
  - And how (functional relationship) do they impact the asset return?

- Model selection (on appropriately chosen data sets) is highly non-trivial, but can be performed in a data-driven manner (instead of using "expert judgement"). Parameter estimation is then typically simple.
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3. Model framework - our simple example

Default model

\[ X_i = r_i Y_{1,2,3} + \sqrt{1 - r_i^2} \cdot \varepsilon_i \quad \text{for all obligors } i \text{ in Group } A, B \text{ and } C \quad (1) \]
\[ r_i \in [0, 1], \quad Y_{1,2,3} \sim N(0, 1) \quad \varepsilon_i \sim \text{NID}(0, 1) \quad \text{and } Y_{1,2,3} \perp \varepsilon_i \quad (2) \]

- Obligors are divided into 3 groups and obligors within the same group are mapped to one systemic market factor \( Y_{1,\ldots,3} \).
- Obligor \( i \in \{1,\ldots,M\} \) defaults at or before time \( \tau_h \) if its normalised (log) asset returns \( X_i \), assumed to follow a standard normal distribution, drop below a default threshold \( d_i = N^{-1}(p_i) \), where \( p_i \) is the default probability for horizon \( \tau_h \).
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3. Model framework - bucketing approach

Determining the loss distribution arising from such model is simple if Monte Carlo simulations are used:

- Simulate asset returns to obtain realizations \( \tilde{X}_i \) from previous slide.
- If \( \tilde{X}_i \leq d_i \), default occurred at time \( \tau_h \).
- The default times \( \tau_i \) of obligor \( i \) are determined via appropriate scaling: \( u_i = \frac{\Phi(\tilde{X}_i)}{p_i} \) and \( \tau_i = u_i \tau_h \).
- The amortization profiles of all loans to obligor \( i \) are then used to determine the size of the loss \( L_i \) to obligor \( i \) and the total portfolio loss \( L = \sum_{i=1}^{M} L_i \).
- For a closed form solution we needed to find an appropriate way to deal with the fact that one obligor \( i \) can have many positions (loans) with different LGDs and amortization profiles.
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- The **amortization profiles** of all loans to obligor $i$ are then used to determine the size of the loss $L_i$ to obligor $i$ and the total portfolio loss $L = \sum_{i=1}^{M} L_i$.
- For a closed form solution we needed to find an appropriate way to deal with the fact that one obligor $i$ can have many positions (loans) with different LGDs and amortization profiles.
Determining the loss distribution arising from such model is simple if Monte Carlo simulations are used:

- Simulate asset returns to obtain realizations $\tilde{X}_i$ from previous slide.
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3. Model framework - bucketing approach

The "Bucketing trick":

<table>
<thead>
<tr>
<th>Loan type</th>
<th>Amount</th>
<th>DtM</th>
<th>Payment schedule</th>
<th>LGD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bullet 1</td>
<td>1M</td>
<td>50</td>
<td>1M in 50 days</td>
<td>0.4</td>
</tr>
<tr>
<td>Bullet 2</td>
<td>1.3M</td>
<td>90</td>
<td>1.3M in 90 days</td>
<td>0.5</td>
</tr>
<tr>
<td>Amortizing</td>
<td>1.5M</td>
<td>365</td>
<td>0.5M in 100d, 0.5M in 230d, 0.5M in 365d</td>
<td>0.4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Bucket 1-122d</th>
<th>Bucket 123-244d</th>
<th>Bucket 245-365d</th>
<th>Loan</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp</td>
<td>Exp × LGD</td>
<td>Exp × LGD</td>
<td>Exp × LGD</td>
<td></td>
</tr>
<tr>
<td>Bullet 1</td>
<td>1M</td>
<td>0.4M</td>
<td>-</td>
<td>1M</td>
</tr>
<tr>
<td>Bullet 2</td>
<td>1.3M</td>
<td>0.65M</td>
<td>-</td>
<td>1.3M</td>
</tr>
<tr>
<td>Amortizing</td>
<td>0.5M</td>
<td>0.2M</td>
<td>0.5M</td>
<td>1.5M</td>
</tr>
<tr>
<td>Sum</td>
<td>2.8M</td>
<td>1.25M</td>
<td>0.5M</td>
<td>3.8M</td>
</tr>
</tbody>
</table>

Eventually, LGDs are adjusted such that Exp of each bucket equals the sum of all cashflows, but Exp × LGD remain the same:

<table>
<thead>
<tr>
<th></th>
<th>Bucket 1-122d</th>
<th>Bucket 123-244d</th>
<th>Bucket 245-365d</th>
<th>Obligor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp</td>
<td>LGD</td>
<td>Exp</td>
<td>LGD</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>2.8M</td>
<td>0.4464</td>
<td>0.5M</td>
<td>0.4</td>
</tr>
</tbody>
</table>

(EIB)

Brown Bag Seminar 18 Sep 2014
The loss function for obligor $i$ is then given by

$$L_i = w_i \sum_{j=1}^{m} \mathbb{1}_{X_i \leq d^{(j)}} Q_i^{(j)}$$

where

- $m$ is the number of buckets
- $w_i$ is the total exposure of obligor $i$ (e.g. 3.8M)
- $\mathbb{1}_{X_i \leq d^{(j)}}$ is the default indicator for the cashflows in bucket $i$ or earlier
- $Q_i^{(j)}$ is the stochastic LGD for cashflows of obligor $i$ in bucket $j$ with mean $\mu_i^{(j)}$ and standard deviation $\sigma_i^{(j)}$

The total portfolio loss is then given by $L = \sum_{i=1}^{M} L_i$.

$ECap$ is chosen to be equal to:

- the **credit VaR** on a targeted confidence level $q$, i.e. the $q$-quantile of $L$ minus the expected loss (EL)
- the **expected shortfall**, i.e. the expected portfolio loss conditional on losses exceeding the $q$-quantile, minus the EL.
3. Model framework - bucketing approach

Few quantities of interest:

- **ECap contributions**, e.g. $ECapC_i$ of the $i$th obligor, are required ex-post in order to break down the overall ECap figure to the individual obligors, products etc. within the portfolio. Desirable: $ECapC_i = w_i \frac{\partial}{\partial w_i} ECap$ and $ECap = \sum ECapC_i$.

- Incremental ECap, IECap, i.e. the difference between the total ECap of the portfolio with and without a **new** loan, is frequently used for pricing purposes on an ex-ante basis.

- **Marginal ECap**, MECap, is the difference between the ECap of the total portfolio with and without an **existing** loan (or obligor).

- IECap and ECapC can be computationally expensive and unstable!

- Closed form approximations can provide accurate, consistent and quick solutions in many cases.
3. Model framework - bucketing approach

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IECap and ECapC can be computationally expensive and unstable!

Closed form approximations can provide accurate, consistent and quick solutions in many cases.
3. Deriving the closed form

The closed form is based on the one derived by [Pykhtin, 2004], a second order Taylor approximation of the $q$-quantile of $L$ around a single R.V. $\bar{L} = L(\bar{Y})$ with suitably chosen standard normal $\bar{Y}$. This approximation is homogeneous of order one in exposures and thus allows for an additive contribution composition. We proceed as follows:

- **Bucketing**: we replace $L_i = w_i \mathbb{1}_{\{x_i \leq d_i\}} Q_i$ in Pykhtin's formula by $L_i = w_i \sum_{j=1}^{m} \mathbb{1}_{\{x_i \leq d_i(j)\}} Q_i^{(j)}$.

- Instead of total VaR/ES, we derive analytical approximations of derivatives $ECapC_i$ directly.

- While Pykhtin's $\bar{Y}$ depends also on exposures, we keep it fixed at the $\bar{Y}$ for the original portfolio. This allows for simple differentiation and a fast two-step implementation procedure:
  - **Step 1**: Slow: most time consuming computations\(^2\).
  - **Step 2**: Fast: compiling all results from the first step by using matrix operations.

---

\(^2\)Pykhtin's formula is actually not fast to compute if the portfolio is large.
3. Deriving the closed form

**Theorem**

A quantile of the portfolio loss at the confidence level $q$ and the corresponding expected shortfall (both before EL subtraction) can be calculated as

$$
t_q(L) = \sum_{i=1}^{M} w_i \frac{\partial}{\partial w_i} t_q(L)
$$

$$
ES_q = \sum_{i=1}^{M} w_i \frac{\partial}{\partial w_i} ES_q \quad \text{with}
$$

$$
\frac{\partial}{\partial w_i} t_q(L) = \frac{\partial}{\partial w_i} t_q(\bar{L}) + \frac{\partial}{\partial w_i} MA_{\infty} + \max \left\{ 0, \frac{\partial}{\partial w_i} GA \right\}
$$

$$
\frac{\partial}{\partial w_i} ES_q = \frac{\partial}{\partial w_i} ES_{q,0} + \left( \frac{\partial}{\partial w_i} ES_q MA_{\infty} + \frac{\partial}{\partial w_i} ES_q GA \right)
$$

*limiting multi-factor adj.*  *granularity adj.*

with quantities outlined in [EIBandMBS, 2014].

---

$a$GA should not be negative, but could be in practice due to approximation errors.
4. A homogeneous test portfolio: description

In the sequel, we compare the closed form formula (CF) with a commercial simulation tool (ST).

- Portfolio: 26 distinct obligors, each one ”copied” 100 times. Hence, 2,600 obligors. Total portfolio size = 10bn€.
- Each obligor belongs to a single Group (e.g. industry sector).
- $\tau_h = 1yr$, $q = 99.9\%$; three systematic risk factors drive the default behavior of all obligors within the given Group.
- Obligors mapped to the same factor carry the same $r_i$.
- Different correlation levels are tested in the sequel.

<table>
<thead>
<tr>
<th></th>
<th>GroupA</th>
<th>GroupB</th>
<th>GroupC</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GroupA</td>
<td>1.0000</td>
<td>$\rho_{AB}$</td>
<td>$\rho_{AC}$</td>
<td>$r_1^2$</td>
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<tr>
<td>GroupB</td>
<td>$\rho_{AB}$</td>
<td>1.0000</td>
<td>$\rho_{BC}$</td>
<td>$r_2^2$</td>
</tr>
<tr>
<td>GroupC</td>
<td>$\rho_{AC}$</td>
<td>$\rho_{BC}$</td>
<td>1.0000</td>
<td>$r_3^2$</td>
</tr>
</tbody>
</table>
4. A homogeneous test portfolio: description

<table>
<thead>
<tr>
<th>Type</th>
<th>Exp (M€)</th>
<th>LGD</th>
<th></th>
</tr>
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<tbody>
<tr>
<td>GroupA</td>
<td>2833</td>
<td>0.07</td>
<td>GroupA</td>
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<tr>
<td>GroupA</td>
<td>881</td>
<td>0.07</td>
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<td>GroupA</td>
<td>828</td>
<td>0.07</td>
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</tr>
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<td>GroupA</td>
<td>540</td>
<td>0.07</td>
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<td>GroupA</td>
<td>521</td>
<td>0.07</td>
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<tr>
<td>GroupA</td>
<td>514</td>
<td>0.07</td>
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<tr>
<td>GroupA</td>
<td>423</td>
<td>0.07</td>
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<tr>
<td>GroupB</td>
<td>363</td>
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<tr>
<td>GroupB</td>
<td>126</td>
<td>0.40</td>
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<tr>
<td>GroupB</td>
<td>252</td>
<td>0.40</td>
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<tr>
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<td>0.40</td>
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<td>GroupB</td>
<td>103</td>
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<td>108</td>
<td>0.50</td>
<td>GroupC</td>
</tr>
</tbody>
</table>

**Table:** Inputs: Exp = exposure, LGD = loss given default and Type = type of the obligors (GroupA, GroupB or GroupC).
4. A homogeneous test portfolio: Total ECap

- Total VaR and ES figures from ST and CF match very well, except when the factor correlations are very low.
- When the factors are independent, CF underestimates VaR and ES due to the inaccuracy in multi-factor adjustment: attention with CF!
- If a small portfolio is chosen (26 obligors without "copying"), the granularity adjustment is 100 times bigger.

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<thead>
<tr>
<th>Factors</th>
<th>ST</th>
<th>CF</th>
<th>1-factor</th>
<th>m adj</th>
<th>GA</th>
<th>EL</th>
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<tr>
<td>High correlations</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>VaR</td>
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**Table:** All figures in M€.
4. A homogeneous test portfolio: Total ECap

- Total VaR and ES figures from ST and CF match very well, except when the factor correlations are very low.
- When the factors are independent, CF underestimates VaR and ES due to the inaccuracy in multi-factor adjustment: attention with CF!
- If a small portfolio is chosen (26 obligors without "copying"), the granularity adjustment is 100 times bigger.

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- ST instability worse for VaR than for ES.

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**Table:** All figures in M€.
4. A homogeneous test portfolio: ECapC and MECap

- All 100 replicated obligors should have identical ECapC and MECap: by construction for CF, but not true in ST which shows instability (in particular for MECap).

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Table: All figures in M€.

(EIB) Brown Bag Seminar 18 Sep 2014 22 / 32
5. A heterogeneous test portfolio: description

- 900 distinct obligors with a total of 3,750 bullet and amortizing loans.
- Loans to one obligor have different maturities and LGD’s.
- \( \tau_h = 3\text{ yrs}, q = 99.97\%, \) 3-factor model with ”high correlation” assumption.
- Low default probabilities, few large exposures. Total portfolio size = 100bn€.

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<th>3-8 months</th>
<th>8-15 months</th>
<th>15-27 months</th>
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<td>6 months</td>
<td>1 year</td>
<td>2 year</td>
<td>3 year</td>
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**Table:** Bucketed cashflows according to the time to payment. Probabilities are scaled starting from 1-year default probabilities using a constant hazard rate model. For low default portfolios, choosing PD points slightly ”towards the right end of the bucket” is reasonably conservative, as shown in separate simulations.
5. A heterogeneous test portfolio: Total ECap

- Total ECap from CF and ST with 30M simulations(!) are very close.
- The choice of buckets leads to a slight over estimation of about 3%.

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<th>VaR</th>
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<th>EL</th>
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<td>0.423%</td>
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</table>

Table: Portfolio ECap and EL figures; figures to be read as percentages of total portfolio size. Computation time for CF:
Step 1 (pre-run) 23mins, Step 2 (compilation) 2.8sec.

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5. A heterogeneous test portfolio: Total ECap

- Total ECap from CF and ST with 30M simulations(!) are very close.
- The choice of buckets leads to a slight over estimation of about 3%.

<table>
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<tbody>
<tr>
<td>CF</td>
<td>7.23%</td>
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5. A heterogeneous test portfolio: ECapC

**Figure:** Relative differences of ECap contributions \( \left( \frac{S_{T} - C_{F}}{S_{T}} \right) \). First: VaRC; Second: ESC. Black circles = GroupA; Red triangles = GroupB; Blue + signs = GroupC. RHS = LHS, but excluding outliers.
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**Figure:** Relative differences of ECap contributions ($\frac{ST - CF}{ST}$). First: VaRC; Second: ESC. Black circles = GroupA; Red triangles = GroupB; Blue + signs = GroupC. RHS = LHS, but excluding outliers.
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Some outliers, i.e. significant differences in ECap contributions between CF and ST, for small exposures:

- Left: CF $>>$ ST for obligors with one loan with few days maturity: no surprise as 2-week PD in first bucket leads to over-estimation of ECapC.

- Left: ST $>>$ CF (100%) for obligors in default: problem in PD scaling in ST for $p_{1Y} = 1$, not a CF approximation problem! ST scales PDs up using constant hazard rate and down linearly, thus under-estimation of EL if loan contains payments before time horizon $\tau_h$.

Once outliers are removed:

- No clear structural difference. Less differences for ES than for VaR.

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5. A heterogeneous test portfolio: IECap

- Consider 10 highest VarC obligors e.g. from GroupB. Add a new loan of 34M€, representing 0.34% of the total portfolio, LGD=0.4, maturity = 3yrs.

- IECap with non-fixed $\tilde{Y}$ (ExactCF) and with fast approximation (Fixed $\tilde{Y}$) are very close, computation time drops from 23min to 2.8sec!

- Linear approx. of IECap via ECapC is widely used in practice and in ST:

  \[
  LIECap_i = w_{i,new} \frac{\mu_{i,new}}{\mu_i} \frac{\partial}{\partial w_i} ECap = w_{i,new} \frac{\mu_{i,new}}{w_i} \frac{\mu_i}{\partial w_i} ECapC_i.
  \]

- LinCF and LinST work fairly well for the 10 obligors, but attention: overall performance is much worse! As all the loan sizes are different, many VaRC’s are negative (not shown in table) for obligors $i$ not contributing to the loss quantile $t_q(L)$:

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  \frac{\partial}{\partial w_i} VaR = \frac{\partial}{\partial w_i} [t_q(L) - EL(L)] = 0 - \frac{\partial}{\partial w_i} EL(L) < 0
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- This leads to negative IVaR, even if the added exposure is very large and true IVaR is big!

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5. A heterogeneous test portfolio: IECap for fast calculations

<table>
<thead>
<tr>
<th>Obligor</th>
<th>Exp</th>
<th>VaRC(CF)</th>
<th>ESC(CF)</th>
<th>EL</th>
<th>Incremental VaR (M€)</th>
<th>Incremental ES (M€)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Existing obligor</td>
<td></td>
<td></td>
<td>New ob.</td>
<td>Existing obligor</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ExactCF</td>
<td>Fixed (\tilde{Y}) CF</td>
<td>LinCF</td>
<td>LinST</td>
<td>ExactCF</td>
</tr>
<tr>
<td>GroupB1</td>
<td>2328.28</td>
<td>245.48</td>
<td>301.30</td>
<td>3.02</td>
<td>6.08</td>
<td>6.10</td>
</tr>
<tr>
<td>GroupB2</td>
<td>3013.45</td>
<td>227.11</td>
<td>299.89</td>
<td>1.79</td>
<td>4.42</td>
<td>4.43</td>
</tr>
<tr>
<td>GroupB3</td>
<td>2011.46</td>
<td>176.12</td>
<td>224.61</td>
<td>1.79</td>
<td>4.90</td>
<td>4.92</td>
</tr>
<tr>
<td>GroupB4</td>
<td>2391.97</td>
<td>109.29</td>
<td>155.32</td>
<td>0.68</td>
<td>2.71</td>
<td>2.72</td>
</tr>
<tr>
<td>GroupB5</td>
<td>1834.55</td>
<td>87.52</td>
<td>122.12</td>
<td>0.63</td>
<td>3.03</td>
<td>3.05</td>
</tr>
<tr>
<td>GroupB6</td>
<td>1584.45</td>
<td>83.61</td>
<td>121.33</td>
<td>0.51</td>
<td>2.57</td>
<td>2.57</td>
</tr>
<tr>
<td>GroupB7</td>
<td>685.30</td>
<td>70.95</td>
<td>87.61</td>
<td>1.00</td>
<td>5.09</td>
<td>5.16</td>
</tr>
<tr>
<td>GroupB8</td>
<td>264.59</td>
<td>60.32</td>
<td>64.89</td>
<td>3.62</td>
<td>9.98</td>
<td>10.29</td>
</tr>
<tr>
<td>GroupB9</td>
<td>258.73</td>
<td>58.30</td>
<td>64.19</td>
<td>2.64</td>
<td>9.39</td>
<td>9.49</td>
</tr>
<tr>
<td>GroupB10</td>
<td>239.78</td>
<td>56.99</td>
<td>60.93</td>
<td>3.70</td>
<td>9.98</td>
<td>10.27</td>
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6. Conclusions

A new closed form formula (CF) for ECapC and IECap in the multi-factor default-mode model based on Pykhtin (2004) was introduced. Performance was tested against a commonly used commercial simulation tool (ST):

- CF yields decent approximation for **Total ECap**, except if risk factor correlations are very low.
- ECapC for large portfolios are very reasonable and within the range of simulation error of ST. Results for ES better than for VaR. Attention with CF for loans with few days to maturity (bucketing).
- CF with fixed $\bar{Y}$ is *fast* and yields good approximation of IECap. Using VaRC for IVaR is dangerous. Using ESC for IES is much safer (whole tail of $L$ taken into account), approximation errors can still be well over 10%.

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M. Pykhtin (2004)
Multi-factor adjustment

Thomas Ribarits, Axel Clement, Heikki Seppälä, Hua Bai and Ser-Huang Poon, (2014)
Economic Capital Modeling: Closed form approximation for real-time applications
Download for free simply by choosing ”Download Anonymously”. 
Thank You
Annex: VAR vs. expected shortfall

Consider a portfolio of 3 obligors $A$, $B$ and $C$ with exposures of 1, 3 and 3.5 EUR, respectively, independent defaults with probabilities of 5%, 5% and 2.5%, respectively:
Annex: VAR vs. expected shortfall

Consider a portfolio of 3 obligors A, B and C with exposures of 1, 3 and 3.5 EUR, respectively, independent defaults with probabilities of 5%, 5% and 2.5%, respectively:

Loss Distribution for a Portfolio of 3 Obligors
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- None / 0
- A / 1
- B / 3
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- AC / 4
- BC / 6.5
- ABC / 7.5

Cumulative Probability

Tail

EL

VaR (99%) /
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![Loss Distribution for a Portfolio of 3 Obligors](image-url)

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