Economic Capital Modeling
Closed form approximation for real-time (pricing) applications

European Investment Bank

t.ribarits@eib.org

Joint work with
Axel Clement, Heikki Seppälä, Jenny Bai Hua and Ser-Huang Poon

resulting from a research cooperation between
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Marie Curie Initial Training Network on Risk Management and Risk Reporting

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Disclaimer

The views and opinions expressed in this presentation are those of the authors and do not necessarily reflect those of the European Investment Bank or the European Investment Bank Institute. All figures shown are based on purely hypothetical test portfolios.

Overview

1. A Primer on Economic Capital

- Economic Capital (ECap) is a *risk metric* designed to measure the riskiness of an entity's (called "Bank" in the sequel) business activities and quantify the level of capital commensurate to such risk.
- As such, it is the output of a *mathematical model*.
- Here, we restrict ourselves to the *credit risk* a Bank accepts as a result of its (contingent) investment activity: credit risk on loans, treasury holdings, counterparty credit risk from derivative activities etc.
- Market risk and operational risk[1], legal risk and reputational risks, while do contribute to a Bank's overall ECap figure, are excluded in this presentation.

[1] These two typically contribute to a relatively small portion of the ECap requirement of a *lending institution* such as EIB.

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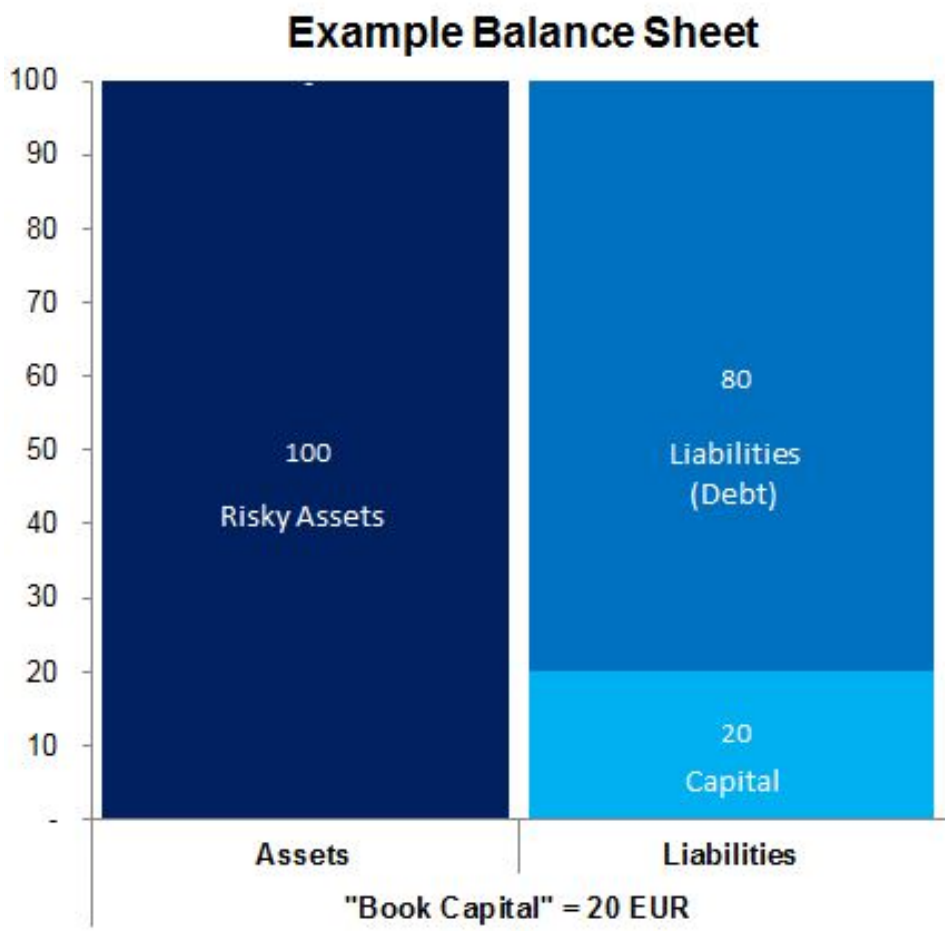
Pillar 2 of the Basle Regulatory Framework requires Banks to perform a so-called *Internal Capital Adequacy Assessment Process (ICAAP)*.



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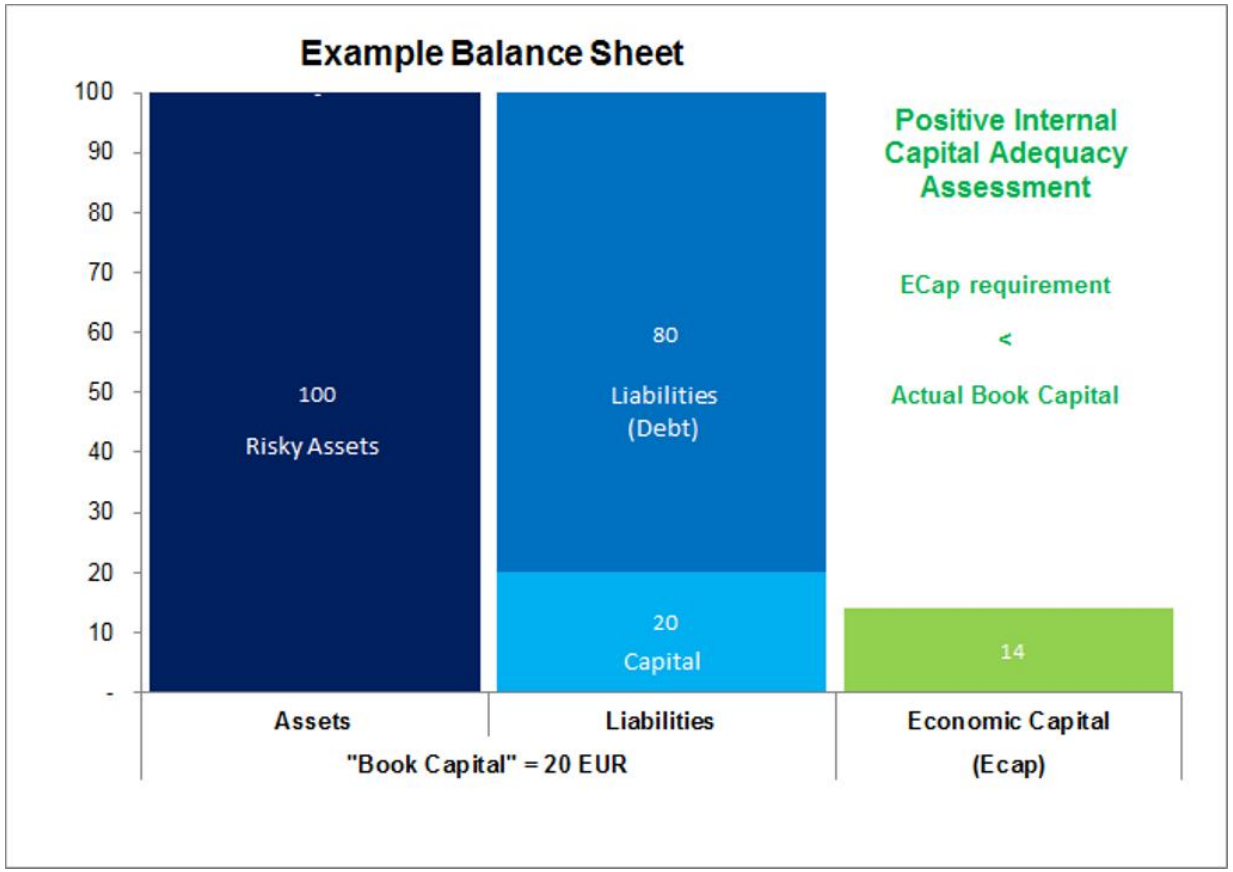
Given a desired confidence level, an example of a sufficiently capitalized Bank looks as follows:



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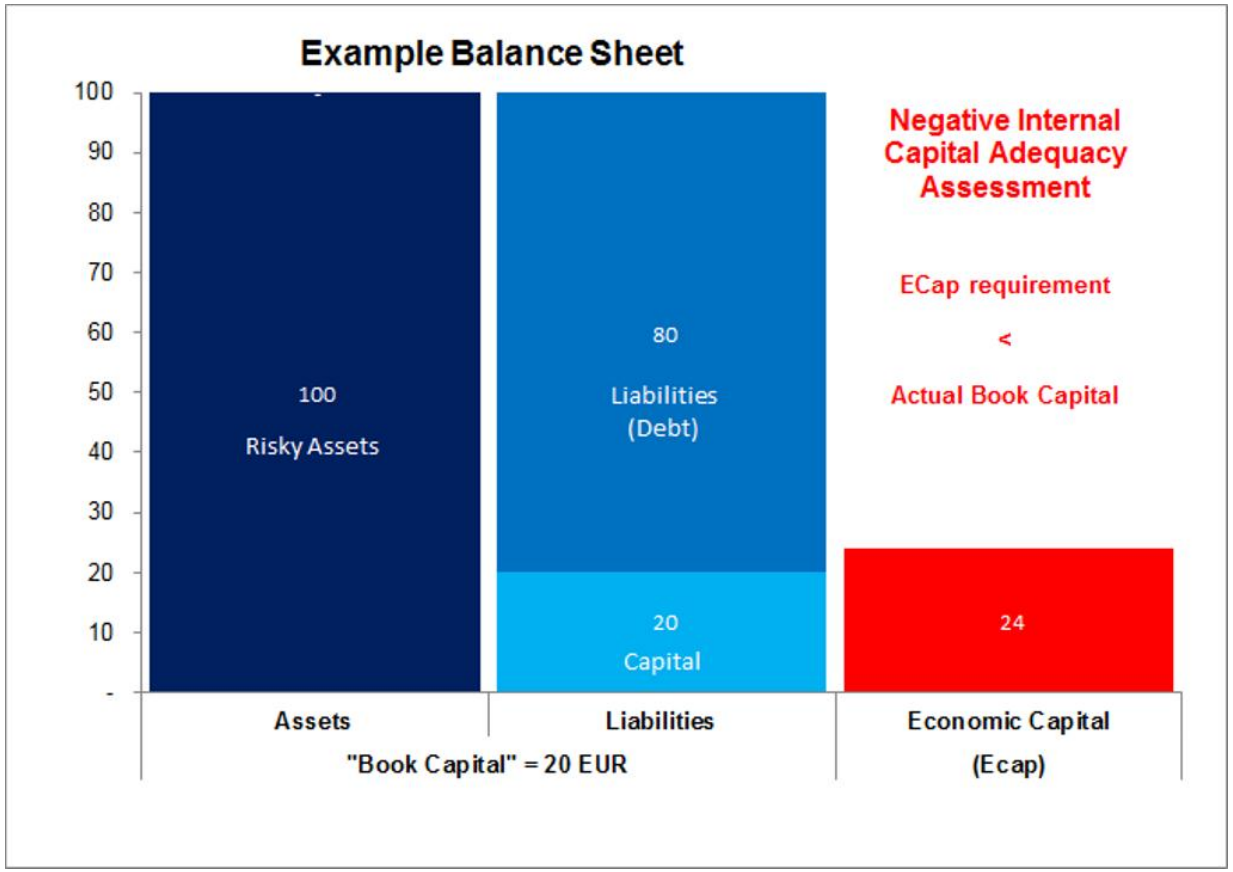
An example of a (for a desired confidence level) not sufficiently capitalized Bank looks as follows:



Back to Model Framework

2. Regulatory vs. Economic Capital

Evidently, there is more than one *mathematical model* for calculating capital requirements.

External view

- **Regulatory Capital** reflects the regulator's view: Pillar 1 rule-based calculation , i.e. "one-size-fits-all formula"
- **Rating Agencies' Capital Figures** reflect their views: typically one methodology for a subset of banks

Internal view

- **Economic Capital** reflects the bank's view: Pillar 2 rule-based calculation or model-based calculation, which is based on the bank's own risk appetite rather than regulatory requirements

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2. Regulatory vs. Economic Capital

Regulatory Capital

The AIRB approach is based upon:

1. Value-at-Risk ("How much capital do we need to survive?")
2. 99.9% confidence level (reflecting an A rating)
3. Time horizon $\tau_h = 1$ year
4. ASRF model with *portfolio invariance* (no concentration risk and no diversification benefits)
5. Only few types of collateral eligible for risk mitigation
6. Default correlation modeling with a single factor driving defaults
7. Possibly still "substitution approach" for guar. exp.

Economic Capital

ECap can e.g. be based upon:

1. Expected shortfall ("What if we do not survive?")
2. Confidence levels more akin to AAA rating
3. Longer time horizon ("How quickly can one counteract a capital inadequacy?")
4. Models based on actual portfolio to reflect concentration risk and diversification benefits
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Time for a first summary

Regulators impose ICAAP as they know well the (restrictive) assumptions underpinning the Pillar 1 regulatory model. Banks have to have an internal model in place and typically use ECap as a **key risk indicator**. Depending on the institution, ECap is applied for:

- Monitoring capital adequacy and portfolio concentrations
- Business planning
- Stress testing
- Ex-post measurement of "risk-adjusted returns" across groups of transactions, business units etc.
- Ex-ante pricing: ECap does address concentration and is, thus, key for unexpected loss pricing.

3. Model framework - general setting

- Estimates for default probabilities (p_i) and "loss-given-default" (LGD) parameters (Q_i) are assumed to be given. But how can we describe the *simultaneous* default behaviour of obligors?
- Merton-based models are inspired by the balance sheet view: obligor i defaults if its asset value ($\bar{X}_i = 100$) drops below a default threshold ($\bar{d}_i = 80$).
- The asset values $\bar{X}_i$ of different obligors may be driven by some common market factors $\bar{Y}_i$ and some un-systematic, obligor-specific components ε_i. Dependence on common market factors introduces **default correlation**.
- All fine, but how do we model the asset values (returns) then?
  1. Asset returns are not observable, which "proxies" to take for X_i?
  2. How many factors drive the asset return X_i of obligor i?
  3. Which factors Y_i are the "right ones" to take?
  4. And how (functional relationship) do they impact the asset return?
- **Model selection** (on appropriately chosen data sets) is highly non-trivial, but can be performed in a **data-driven** manner (instead of using "expert judgement"). Parameter estimation is then typically simple.

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Default model

$$X_i = r_i Y_{1,2,3} + \sqrt{1 - r_i^2} \cdot \varepsilon_i \quad \text{for all obligors } i \text{ in Group } A, B \text{ and } C \tag{1}$$

$$r_i \in [0,1], \quad Y_{1,2,3} \sim N(0,1) \quad \varepsilon_i \sim NID(0,1) \quad and Y_{1,2,3} \perp \varepsilon_i \tag{2}$$

- Obligors are divided into 3 groups and obligors within the same group are mapped to one systemic market factor $Y_1, \ldots, Y_3$.
- Obligor $i \in \{1, ..., M\}$ defaults at or before time τ_h if its normalised (log) asset returns X_i, assumed to follow a *standard normal* distribution, drop below a default threshold $d_i = N^{-1}(p_i)$, where p_i is the default probability for horizon τ_h.

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Determining the loss distribution arising from such model is simple if Monte Carlo simulations are used:

- Simulate asset returns to obtain realizations $\tilde{X}_i$ from previous slide.
- If $\tilde{X}_i \leq d_i$, default occured at time τ_h.
- The **default times** τ_i of obligor i are determined via appropriate scaling: $u_i = \frac{\Phi(\tilde{X}_i)}{p_i}$ and $\tau_i = u_i \tau_h$.
- The **amortization profiles** of all loans to obligor i are then used to determine the size of the loss L_i to obligor i and the total portfolio loss $L = \sum_{i=1}^{M} L_i$.
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- The **default times** τ_i of obligor i are determined via appropriate scaling: $u_i = \frac{\Phi(\tilde{X}_i)}{p_i}$ and $\tau_i = u_i \tau_h$.
- The **amortization profiles** of all loans to obligor i are then used to determine the size of the loss L_i to obligor i and the total portfolio loss $L = \sum_{i=1}^{M} L_i$.
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3. Model framework - bucketing approach

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3. Model framework - bucketing approach

The "Bucketing trick":

Loan type	Amount	DtM	Payment schedule	LGD
Bullet 1	1M	50	1M in 50 days	0.4
Bullet 2	1.3M	90	1.3M in 90 days	0.5
Amortizing	1.5M	365	0.5M in 100d, 0.5M in 230d, 0.5M in 365d	0.4

	Bucket 1-122d		Bucket 123-244d		Bucket 245-365d		Loan	
	Exp	E×LGD	Exp	E×LGD	Exp	E×LGD	Exp	E×LGD
Bullet 1	1M	0.4M	-	-	-	-	1M	0.4M
Bullet 2	1.3M	0.65M	-	-	-	-	1.3M	0.65M
Amortizing	0.5M	0.2M	0.5M	0.2M	0.5M	0.2M	1.5M	0.6M
Sum	2.8M	1.25M	0.5M	0.2M	0.5M	0.2M	3.8M	1.65M

	Bucket 1-122d		Bucket 123-244d		Bucket 245-365d		Obligor	
	Exp	LGD	Exp	LGD	Exp	LGD	Exp	LGD
Total	2.8M	0.4464	0.5M	0.4	0.5M	0.4	3.8M	0.4342105

Eventually, LGDs are adjusted such that Exp of each bucket equals the sum of all cashflows, but Exp $\times$ LGD remain the same:

	Bucket 1-122d		Bucket 123-244d		Bucket 245-365d		Obligor	
	Exp	LGD	Exp	LGD	Exp	LGD	Exp	LGD
Total	3.8M	0.3289	3.8M	0.05263	3.8M	0.05263	3.8M	0.4342105

3. Model framework - bucketing approach

- The loss function for obligor i is then given by $L_i = w_i \sum_{j=1}^{m} \mathbb{1}_{X_i \leq d_i^{(j)}} Q_i^{(j)}$ where
  - m is the number of buckets
  - w_i is the total exposure of obligor i (e.g. 3.8M)
  - $\mathbb{1}_{X_i \leq d_i^{(j)}}$ is the default indicator for the cashflows in bucket i or earlier
  - $Q_i^{(j)}$ is the stochastic LGD for cashflows of obligor i in bucket j with mean $\mu_i^{(j)}$ and standard deviation $\sigma_i^{(j)}$
- The total portfolio loss is then given by $L = \sum_{i=1}^{M} L_i$.
- *ECap* is chosen to be equal to:
  - the **credit VaR** on a targeted confidence level q, i.e. the q-quantile of L *minus* the expected loss (EL)
  - the **expected shortfall**, i.e. the expected portfolio loss conditional on losses exceeding the q-quantile, *minus* the EL.

3. Model framework - bucketing approach

Few quantities of interest:

- **ECap contributions**, e.g. $ECapC_i$ of the ith obligor, are required ex-post in order to break down the overall ECap figure to the individual obligors, products etc. within the portfolio. Desirable: $ECapC_i = w_i \frac{\partial}{\partial w_i} ECap$ and $ECap = \sum ECapC_i$.
- **Incremental ECap**, IECap, i.e. the difference between the total ECap of the portfolio with and without a new loan, is frequently used for pricing purposes on an ex-ante basis.
- **Marginal ECap**, MECap, is the difference between the ECap of the total portfolio with and without an existing loan (or obligor).
- IECap and ECapC can be computationally expensive and unstable!
- Closed form approximations can provide accurate, consistent and quick solutions in many cases.

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3. Deriving the closed form

The closed form is based on the one derived by [Pykhtin, 2004], a second order Taylor approximation of the q-quantile of L around a single R.V. $\bar{L} = L(\bar{Y})$ with suitably chosen standard normal $\bar{Y}$. This approximation is homogeneous of order one in exposures and thus allows for an additive contribution composition. We proceed as follows:

- Bucketing: we replace $L_i = w_i \mathbb{1}_{\{X_i \leq d_i\}} Q_i$ in Pykhtin's formula by $L_i = w_i \sum_{j=1}^{m} \mathbb{1}_{\left\{X_i \leq d_i^{(j)}\right\}} Q_i^{(j)}$.
- Instead of total VaR/ES, we derive analytical approximations of *derivatives* $ECapC_i$ directly.
- While Pykhtin's $\bar{Y}$ depends also on exposures, we keep it fixed at the $\bar{Y}$ for the original portfolio. This allows for simple differentiation and a fast two-step implementation procedure:
  - Step 1: Slow: most time consuming computations[2].
  - Step 2: Fast: compiling all results from the first step by using matrix operations.

[2] Pykhtin's formula is actually not fast to compute if the portfolio is large.

3. Deriving the closed form

Theorem

A quantile of the portfolio loss at the confidence level q and the corresponding expected shortfall (both before EL subtraction) can be calculated as

$$t_q(L) = \sum_{i=1}^{M} w_i \frac{\partial}{\partial w_i} t_q(L) \qquad \mathrm{ES}_q = \sum_{i=1}^{M} w_i \frac{\partial}{\partial w_i} \mathrm{ES}_q \quad \textit{with}$$

$$\begin{aligned}
\frac{\partial}{\partial w_i} t_q(L) &= \frac{\partial}{\partial w_i} t_q(\bar{L}) + \frac{\partial}{\partial w_i} \mathrm{MA}_\infty + \max\left\{0, \frac{\partial}{\partial w_i} \mathrm{GA}\right\}^{a} \\
\frac{\partial}{\partial w_i} \mathrm{ES}_q &= \frac{\partial}{\partial w_i} \mathrm{ES}_{q,0} + \underbrace{\frac{\partial}{\partial w_i} \mathrm{ES}_q \, \mathrm{MA}_\infty}_{\textit{limiting multi-factor adj.}} + \underbrace{\frac{\partial}{\partial w_i} \mathrm{ES}_q \, \mathrm{GA}}_{\textit{granularity adj.}}
\end{aligned}$$

with quantities outlined in [EIBandMBS, 2014].

[a] GA should not be negative, but could be in practice due to approximation errors.

4. A homogeneous test portfolio: description

In the sequel, we compare the closed form formula (CF) with a commercial simulation tool (ST).

- Portfolio: 26 distinct obligors, each one "copied" 100 times. Hence, 2,600 obligors. Total portfolio size = 10bn€ .
- Each obligor belongs to a single Group (e.g. industry sector).
- $\tau_h = 1yr$, $q = 99.9\%$; three systematic risk factors drive the default behavior of all obligors within the given Group.
- Obligors mapped to the same factor carry the same r_i.
- Different correlation levels are tested in the sequel.

	GroupA	GroupB	GroupC	R^2
GroupA	1.0000	ρ_{AB}	ρ_{AC}	r_1^2
GroupB	ρ_{AB}	1.0000	ρ_{BC}	r_2^2
GroupC	ρ_{AC}	ρ_{BC}	1.0000	r_3^2

4. A homogeneous test portfolio: description

	Exp (M€)	LGD	Type
GroupA1	2833	0.07	GroupA
GroupA2	881	0.07	GroupA
GroupA3	828	0.07	GroupA
GroupA4	540	0.07	GroupA
GroupA5	521	0.07	GroupA
GroupA6	514	0.07	GroupA
GroupA7	423	0.07	GroupA
GroupB1	363	0.40	GroupB
GroupB2	126	0.40	GroupB
GroupB3	252	0.40	GroupB
GroupB4	107	0.40	GroupB
GroupB5	522	0.40	GroupB
GroupB6	225	0.40	GroupB
GroupB7	85	0.40	GroupB
GroupB8	190	0.40	GroupB
GroupB9	103	0.40	GroupB
GroupC1	278	0.50	GroupC
GroupC2	173	0.50	GroupC
GroupC3	141	0.50	GroupC
GroupC4	139	0.50	GroupC
GroupC5	138	0.50	GroupC
GroupC6	136	0.50	GroupC
GroupC7	130	0.50	GroupC
GroupC8	128	0.50	GroupC
GroupC9	117	0.50	GroupC
GroupC10	108	0.50	GroupC

Table: Inputs: Exp = exposure, LGD = loss given default and Type = type of the obligors (GroupA, GroupB or GroupC).

4. A homogeneous test portfolio: Total ECap

- Total VaR and ES figures from ST and CF match very well, except when the factor correlations are very low.
- When the factors are independent, CF underestimates VaR and ES due to the inaccuracy in multi-factor adjustment: attention with CF!
- If a small portfolio is chosen (26 obligors without "copying"), the granularity adjustment is 100 times bigger.

Factors		ST	CF	1-factor	m adj	GA	EL
High correlations	VaR	153.56	151.97	149.21	4.35	0.91	2.50
	ES	240.86	239.3	235.91	4.86	1.04	2.50
Independent (zero correlations)	VaR	140.62	111.68	56.27	56.72	1.19	2.50
	ES	189.67	151.42	80.93	71.66	1.33	2.50
One factor (perfect correlation)	VaR	181.70	180.38	182.04	0.00	0.84	2.50
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Table: All figures in M€.

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- All 100 replicated obligors should have identical ECapC and MECap: by construction for CF, but not true in ST which shows instability (in particular for MECap).
- ST instability worse for VaR than for ES.

Inputs		Marginal				Contributions							
	Exp	CFVaR	STVaR mean	min	max	CFVaR	STVaR (mean)	min	max	CFES	STES mean	min	max
Total	10,000	148.76	145.65	64.19	258.63	**151.97**	153.48	141.21	166.73	**239.30**	241.28	235.53	247.83
GroupA1	2832.70	8.66	10.36	6.30	13.88	8.82	10.30	9.37	11.66	12.22	13.00	12.55	13.60
GroupA2	881.30	0.31	0.06	-0.01	0.36	0.31	0.37	0.27	0.49	0.50	0.65	0.59	0.70
GroupA3	827.80	1.41	0.79	-0.03	2.84	1.40	1.68	1.44	1.98	2.04	2.30	2.15	2.41
GroupA4	540.00	1.24	0.50	-0.05	2.00	1.23	1.46	1.25	1.64	1.75	1.93	1.85	2.00
GroupA5	520.70	0.18	0.04	0.00	0.30	0.18	0.23	0.14	0.29	0.29	0.38	0.33	0.42
GroupA6	514.00	3.69	3.21	0.19	8.13	3.69	4.26	3.96	4.59	4.81	4.74	4.63	4.87
GroupA7	423.30	4.50	4.67	0.15	8.55	4.50	5.12	4.87	5.41	5.69	5.38	5.29	5.51
GroupB1	362.70	5.40	6.84	1.49	8.86	5.33	6.26	5.50	7.02	10.41	13.60	13.22	14.06
GroupB2	125.70	1.43	0.84	0.01	2.50	1.40	1.69	1.49	1.91	2.87	3.94	3.75	4.05
GroupB3	252.00	4.84	6.14	1.55	8.85	4.81	5.71	5.01	6.33	8.95	11.23	10.96	11.60
GroupB4	107.40	3.57	2.94	0.34	7.09	3.57	4.11	3.81	4.45	6.00	6.93	6.79	7.11
GroupB5	521.70	29.53	31.34	23.36	41.20	31.07	34.84	33.03	36.97	46.89	48.85	48.00	49.67
GroupB6	225.00	9.90	11.07	8.30	14.18	10.04	11.45	10.76	12.65	16.00	17.51	17.20	17.95
GroupB7	85.00	2.09	1.56	0.00	6.29	2.08	2.42	2.22	2.71	3.70	4.47	4.37	4.56
GroupB8	189.70	8.35	9.46	7.02	13.01	8.46	9.70	9.11	10.20	13.48	14.77	14.53	15.13
GroupB9	102.70	4.53	5.12	1.55	8.84	4.56	5.21	4.81	5.43	7.28	7.99	7.86	8.15
GroupC1	277.70	13.24	13.42	8.71	18.67	13.83	11.16	10.13	12.05	21.36	18.16	17.76	18.71
GroupC2	172.80	4.72	3.93	0.35	10.42	4.76	3.75	3.37	4.08	8.13	7.29	7.05	7.58
GroupC3	140.80	5.14	4.11	0.94	11.30	5.23	4.21	3.89	4.67	8.49	7.44	7.18	7.69
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GroupC8	128.40	6.19	5.43	0.62	11.44	6.35	5.16	4.69	5.59	9.82	8.39	8.19	8.65
GroupC9	116.90	3.20	1.98	0.00	8.26	3.21	2.56	2.26	2.93	5.49	4.95	4.79	5.08
GroupC10	108.40	5.24	4.25	0.38	8.61	5.35	4.36	4.03	4.78	8.28	7.08	6.93	7.30

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GroupC1	277.70	13.24	13.42	8.71	18.67	13.83	11.16	10.13	12.05	21.36	18.16	17.76	18.71
GroupC2	172.80	4.72	3.93	0.35	10.42	4.76	3.75	3.37	4.08	8.13	7.29	7.05	7.58
GroupC3	140.80	5.14	4.11	0.94	11.30	5.23	4.21	3.89	4.67	8.49	7.44	7.18	7.69
GroupC4	139.40	5.09	4.35	0.35	10.26	5.18	4.14	3.72	4.58	8.40	7.34	7.19	7.55
GroupC5	137.60	5.03	3.84	0.51	9.15	5.11	4.10	3.64	4.38	8.29	7.26	7.05	7.41
GroupC6	136.50	4.99	4.10	0.29	12.15	5.07	4.05	3.65	4.44	8.23	7.20	7.06	7.36
GroupC7	130.00	6.27	5.28	1.87	11.49	6.43	5.20	4.79	5.50	9.94	8.50	8.26	8.71
GroupC8	128.40	6.19	5.43	0.62	11.44	6.35	5.16	4.69	5.59	9.82	8.39	8.19	8.65
GroupC9	116.90	3.20	1.98	0.00	8.26	3.21	2.56	2.26	2.93	5.49	4.95	4.79	5.08
GroupC10	108.40	5.24	4.25	0.38	8.61	5.35	4.36	4.03	4.78	8.28	7.08	6.93	7.30

Table: All figures in M€.

4. A homogeneous test portfolio: ECapC and MECap

- All 100 replicated obligors should have identical ECapC and MECap: by construction for CF, but not true in ST which shows instability (in particular for MECap).
- ST instability worse for VaR than for ES.

Inputs		Marginal				Contributions							
	Exp	CFVaR	STVaR mean	min	max	CFVaR	STVaR (mean)	min	max	CFES	STES mean	min	max
Total	10,000	148.76	145.65	64.19	258.63	**151.97**	153.48	141.21	166.73	**239.30**	241.28	235.53	247.83
GroupA1	2832.70	8.66	10.36	6.30	13.88	8.82	10.30	9.37	11.66	12.22	13.00	12.55	13.60
GroupA2	881.30	0.31	0.06	-0.01	0.36	0.31	0.37	0.27	0.49	0.50	0.65	0.59	0.70
GroupA3	827.80	1.41	0.79	-0.03	2.84	1.40	1.68	1.44	1.98	2.04	2.30	2.15	2.41
GroupA4	540.00	1.24	0.50	-0.05	2.00	1.23	1.46	1.25	1.64	1.75	1.93	1.85	2.00
GroupA5	520.70	0.18	0.04	0.00	0.30	0.18	0.23	0.14	0.29	0.29	0.38	0.33	0.42
GroupA6	514.00	3.69	3.21	0.19	8.13	3.69	4.26	3.96	4.59	4.81	4.74	4.63	4.87
GroupA7	423.30	4.50	4.67	0.15	8.55	4.50	5.12	4.87	5.41	5.69	5.38	5.29	5.51
GroupB1	362.70	5.40	6.84	1.49	8.86	5.33	6.26	5.50	7.02	10.41	13.60	13.22	14.06
GroupB2	125.70	1.43	0.84	0.01	2.50	1.40	1.69	1.49	1.91	2.87	3.94	3.75	4.05
GroupB3	252.00	4.84	6.14	1.55	8.85	4.81	5.71	5.01	6.33	8.95	11.23	10.96	11.60
GroupB4	107.40	3.57	2.94	0.34	7.09	3.57	4.11	3.81	4.45	6.00	6.93	6.79	7.11
GroupB5	521.70	29.53	31.34	23.36	41.20	31.07	34.84	33.03	36.97	46.89	48.85	48.00	49.67
GroupB6	225.00	9.90	11.07	8.30	14.18	10.04	11.45	10.76	12.65	16.00	17.51	17.20	17.95
GroupB7	85.00	2.09	1.56	0.00	6.29	2.08	2.42	2.22	2.71	3.70	4.47	4.37	4.56
GroupB8	189.70	8.35	9.46	7.02	13.01	8.46	9.70	9.11	10.20	13.48	14.77	14.53	15.13
GroupB9	102.70	4.53	5.12	1.55	8.84	4.56	5.21	4.81	5.43	7.28	7.99	7.86	8.15
GroupC1	277.70	13.24	13.42	8.71	18.67	13.83	11.16	10.13	12.05	21.36	18.16	17.76	18.71
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Table: All figures in M€.

5. A heterogeneous test portfolio: description

- 900 distinct obligors with a total of $3,750$ bullet and amortizing loans.
- Loans to one obligor have different maturities and LGD's.
- $\tau_h = 3yrs$, $q = 99.97\%$, 3-factor model with "high correlation" assumption.
- Low default probabilities, few large exposures. Total portfolio size = 100bn€ .

Bucket	0-3 weeks	3 weeks - 3 months	3-8 months	8-15 months	15-27 months	27+ months
PD	2 weeks	2 months	6 months	1 year	2 year	3 year

Table: Bucketed cashflows according to the time to payment. Probabilities are scaled starting from 1-year default probabilities using a constant hazard rate model. For low default portfolios, choosing *PD* points slightly "towards the right end of the bucket" is reasonably conservative, as shown in separate simulations.

5. A heterogeneous test portfolio: Total ECap

- Total ECap from CF and ST with 30M simulations(!)[3] are very close.
- The choice of buckets leads to a slight over estimation of about 3%.

Method	VaR	ES	EL
CF	7.23%	8.80%	0.427%
ST	7.00%	8.53%	0.423%

Table: Portfolio ECap and EL figures; figures to be read as percentages of total portfolio size. Computation time for CF: Step 1 (pre-run) 23mins, Step 2 (compilation) 2.8sec.

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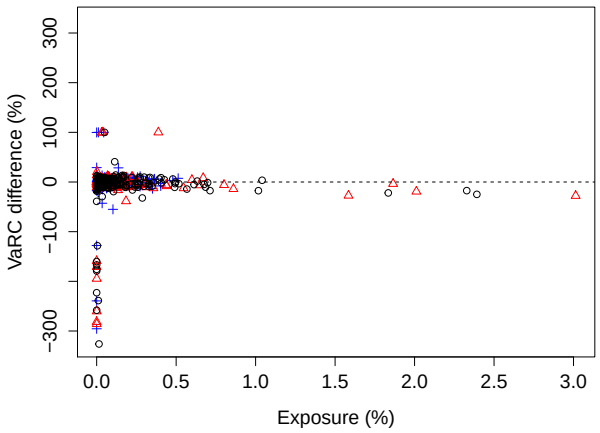
5. A heterogeneous test portfolio: ECapC



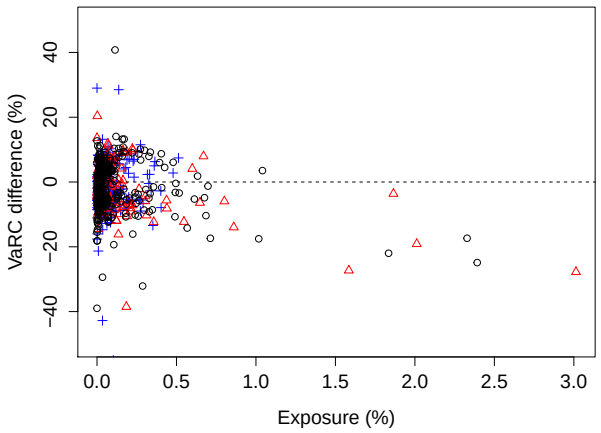


Figure: Relative differences of ECap contributions ($\frac{\text{ST}-\text{CF}}{\text{ST}}$). First: VaRC; Second: ESC. Black circles = GroupA; Red triangles = GroupB; Blue +-signs = GroupC. RHS = LHS, but excluding outliers.

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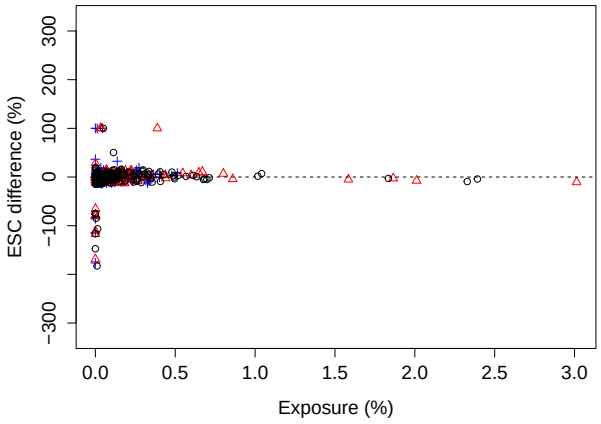

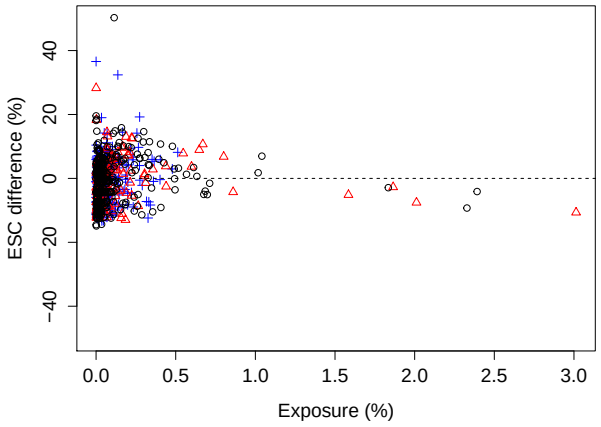

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Some outliers, i.e. significant differences in ECap contributions between CF and ST, for small exposures:

- Left: CF >> ST for obligors with one loan with few days maturity: no surprise as 2-week PD in first bucket leads to over-estimation of ECapC.
- Left: ST >> CF (100%) for obligors in default: problem in PD scaling in ST for $p_{1Y} = 1$, not a CF approximation problem! ST scales PDs up using const. hazard rate and down linearly, thus under-estimation of EL if loan contains payments before time horizon τ_h.

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5. A heterogeneous test portfolio: IECap

- Consider 10 highest VarC obligors e.g. from GroupB. Add a new loan of 34M€, representing 0.34% of the total portfolio, LGD=0.4, maturity = 3yrs.
- IECap with non-fixed $\bar{Y}$ (ExactCF) and with fast approximation (Fixed $\bar{Y}$) are very close, computation time drops **from 23min to 2.8sec**!
- Linear approx. of IECap via ECapC is widely used in practice and in ST:

$$\text{LIECap}_i = w_{i,\text{new}} \frac{\mu_{i,\text{new}}}{\mu_i} \frac{\partial}{\partial w_i} \text{ECap} = \frac{w_{i,\text{new}}}{w_i} \frac{\mu_{i,\text{new}}}{\mu_i} ECapC_i.$$

- LinCF and LinST work fairly well for the 10 obligors, but attention: **overall performance is much worse!** As all the loan sizes are different, many VaRC's are negative (not shown in table) for obligors i not contributing to the loss quantile $t_q(L)$:

$$\frac{\partial}{\partial w_i} \text{VaR} = \frac{\partial}{\partial w_i} [t_q(L) - \text{EL}(L)] = 0 - \frac{\partial}{\partial w_i} \text{EL}(L) < 0$$

- This leads to negative IVaR, even if the added exposure is very large and true IVaR is big!
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5. A heterogeneous test portfolio: IECap for fast calculations

					Incremental VaR (M€)					Incremental ES (M€)				
Obligor	Exp	VaRC(CF)	ESC(CF)	EL	Existing obligor				New ob.	Existing obligor				New ob.
					ExactCF	Fixed $\bar{Y}$ CF	LinCF	LinST	ExactCF	ExactCF	Fixed $\bar{Y}$ CF	LinCF	LinST	Exact CF
GroupB1	2328.28	245.48	301.30	3.02	6.08	6.10	5.43	4.85	4.80	7.48	7.49	6.67	6.37	6.09
GroupB2	3013.45	227.11	299.89	1.79	4.42	4.43	3.81	3.10	3.20	5.84	5.84	5.03	4.74	4.35
GroupB3	2011.46	176.12	224.61	1.79	4.90	4.92	4.40	3.79	3.95	6.28	6.29	5.60	5.28	5.18
GroupB4	2391.97	109.29	155.32	0.68	2.71	2.72	2.40	2.00	2.11	3.88	3.89	3.42	3.44	3.06
GroupB5	1834.55	87.52	122.12	0.63	3.03	3.05	2.48	2.12	2.56	4.22	4.24	3.46	3.50	3.60
GroupB6	1584.45	83.61	121.33	0.51	2.57	2.57	2.03	1.72	2.11	3.70	3.70	2.94	2.92	3.06
GroupB7	685.30	70.95	87.61	1.00	5.09	5.16	5.02	4.59	4.80	6.40	6.46	6.20	6.03	6.09
GroupB8	264.59	60.32	64.89	3.62	9.98	10.29	9.94	9.84	10.27	10.69	10.99	10.70	9.96	11.03
GroupB9	258.73	58.30	64.19	2.64	9.39	9.49	8.92	8.75	9.39	10.25	10.33	9.82	9.23	10.36
GroupB10	239.78	56.99	60.93	3.70	9.98	10.27	10.24	10.41	10.27	10.71	10.99	10.94	10.29	11.03

Table: Ten GroupB obligors with the highest VaR contributions. IVaR and IES of an additional 34M€ loan from CF run twice (ExactCF), CF with fast approximation (Fixed $\bar{Y}CF$), linear approximations based on VarC and ESC (LinCF, LinST), "exact" IVaR and IES of a new obligor with identical characteristics (New ob. ExactCF). All figures in M€.

6. Conclusions

A new closed form formula (CF) for ECapC and IECap in the multi-factor default-mode model based on Pykhtin (2004) was introduced. Performance was tested against a commonly used commercial simulation tool (ST):

- CF yields decent approximation for **Total ECap**, except if risk factor correlations are very low.
- **ECapC** for large portfolios are very reasonable and within the range of simulation error of ST[4]. Results for ES better than for VaR. Attention with CF for loans with few days to maturity (bucketing).
- CF with fixed $\bar{Y}$ is *fast* and yields good approximation of **IECap**. Using VaRC for IVaR is dangerous. Using ESC for IES is much safer (whole tail of L taken into account), approximation errors can still be well over 10%.

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Economic Capital Modeling: Closed form approximation for real-time applications

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Thank You

Annex: VAR vs. expected shortfall

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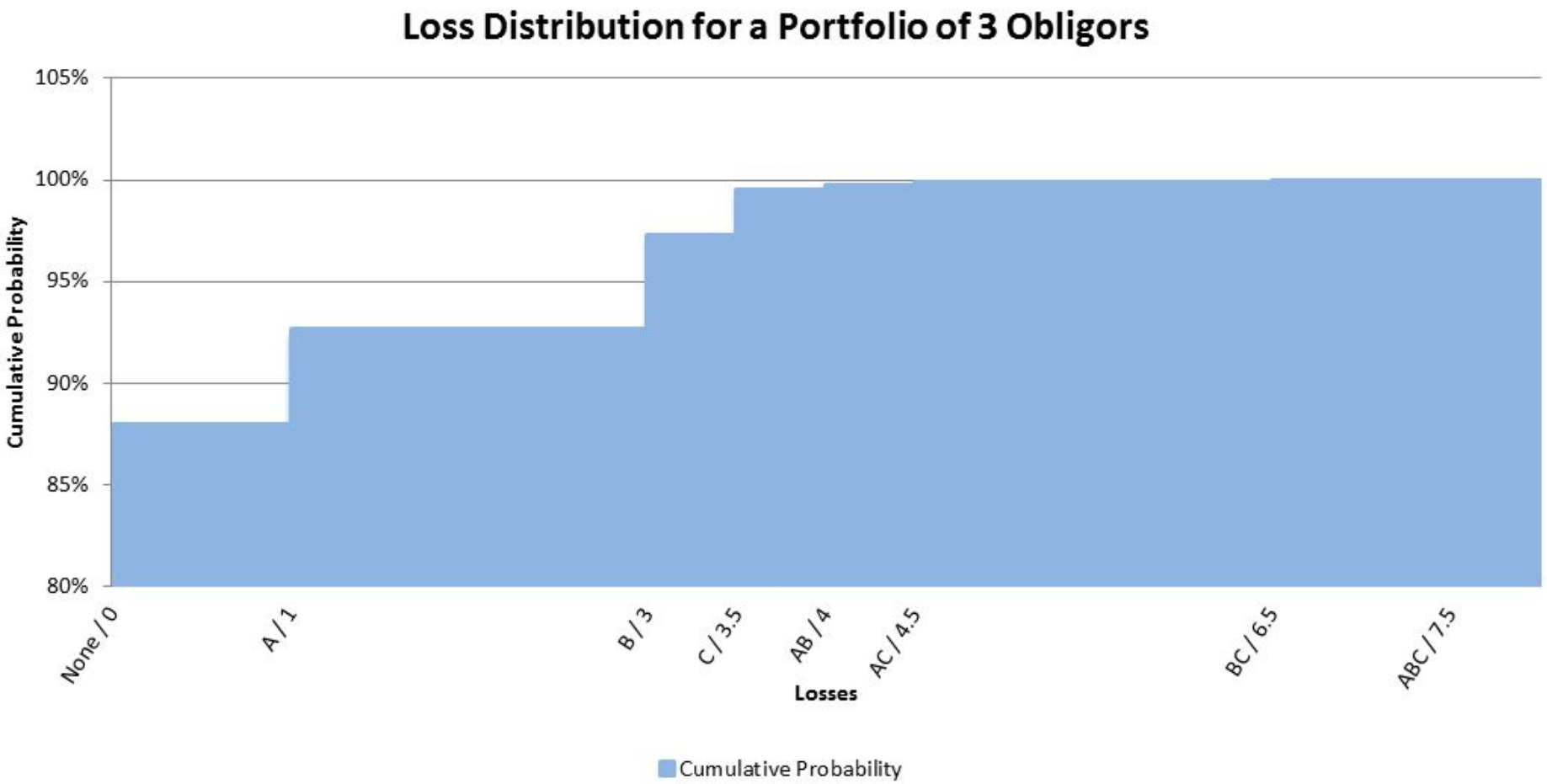

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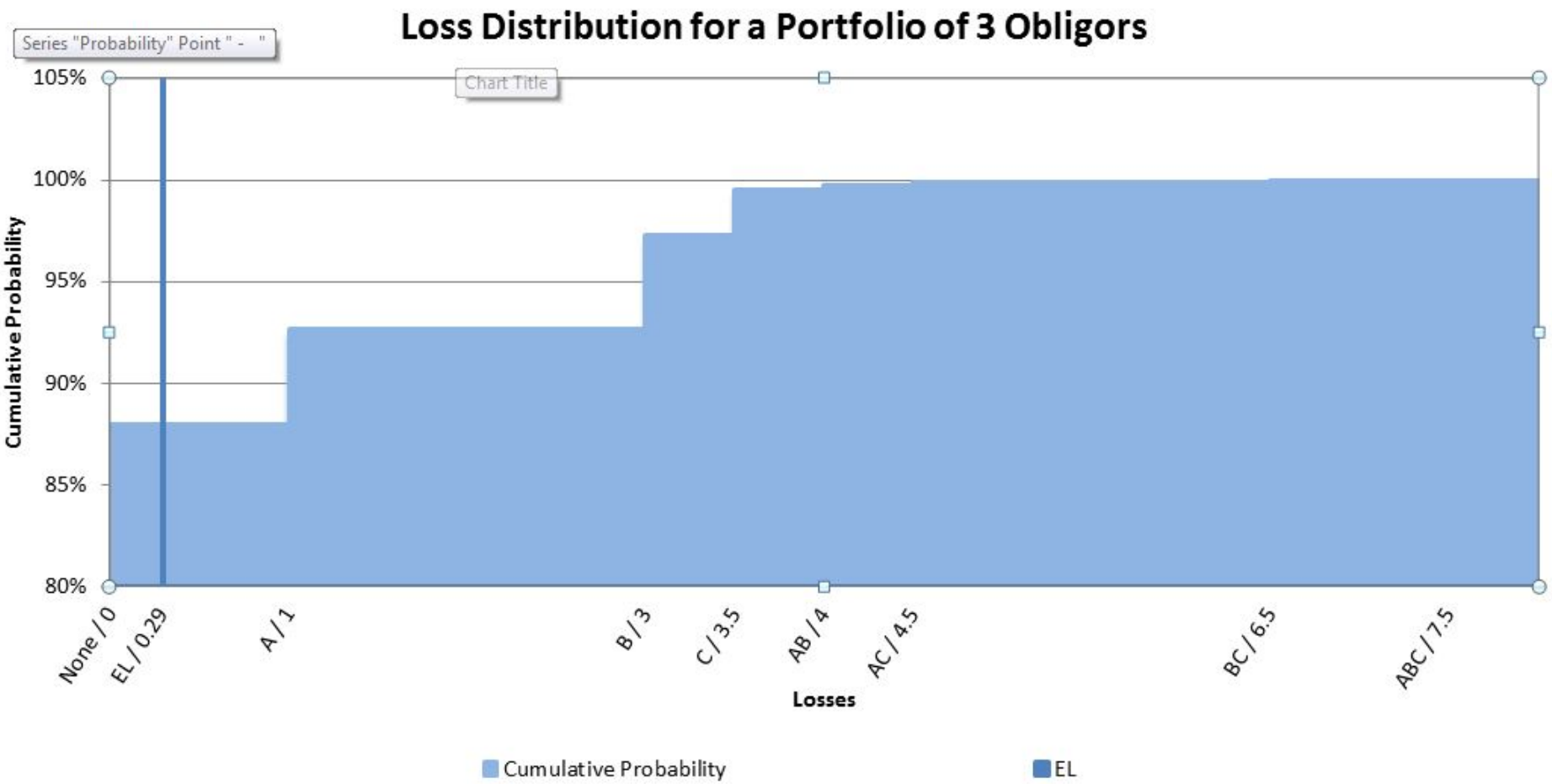

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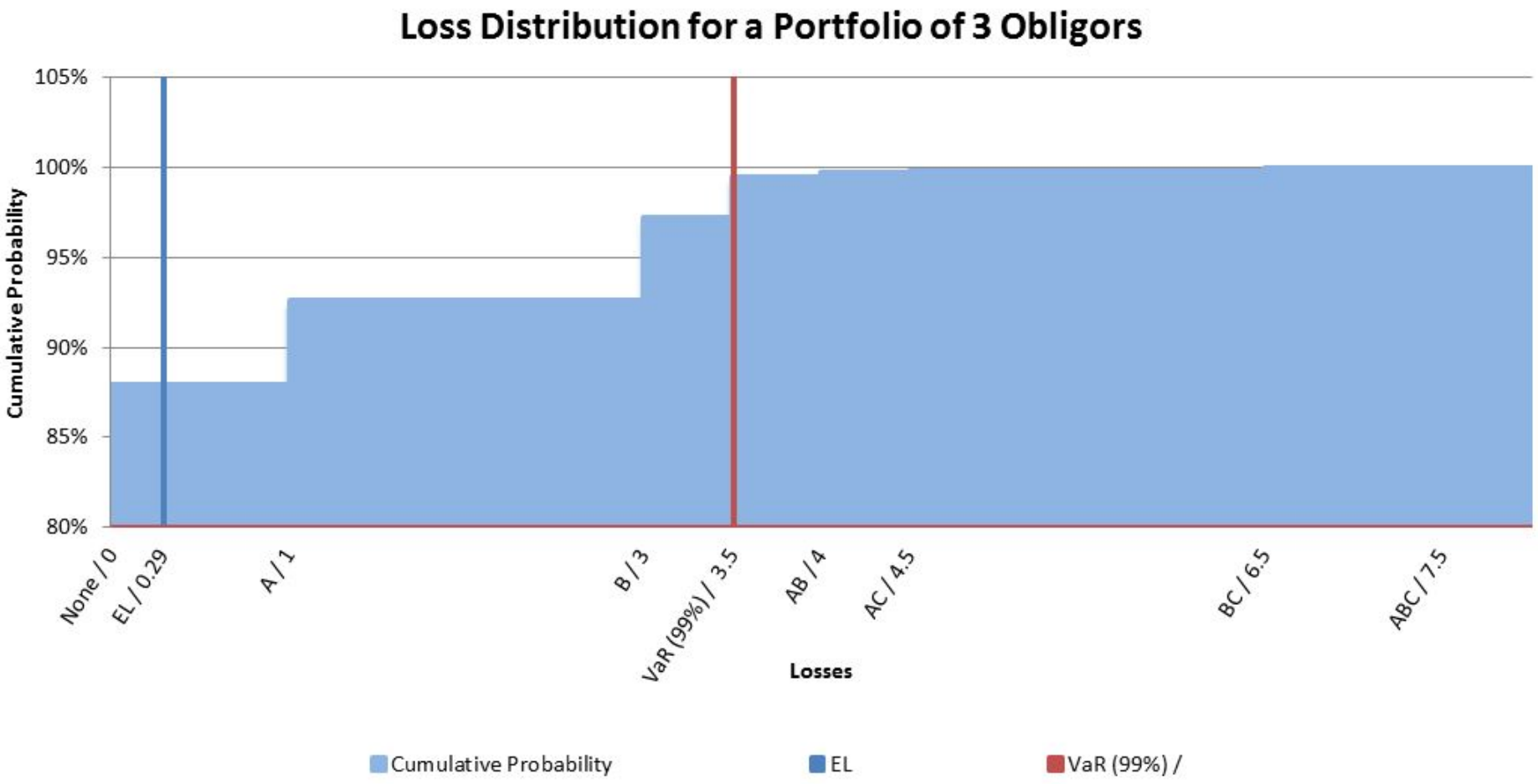

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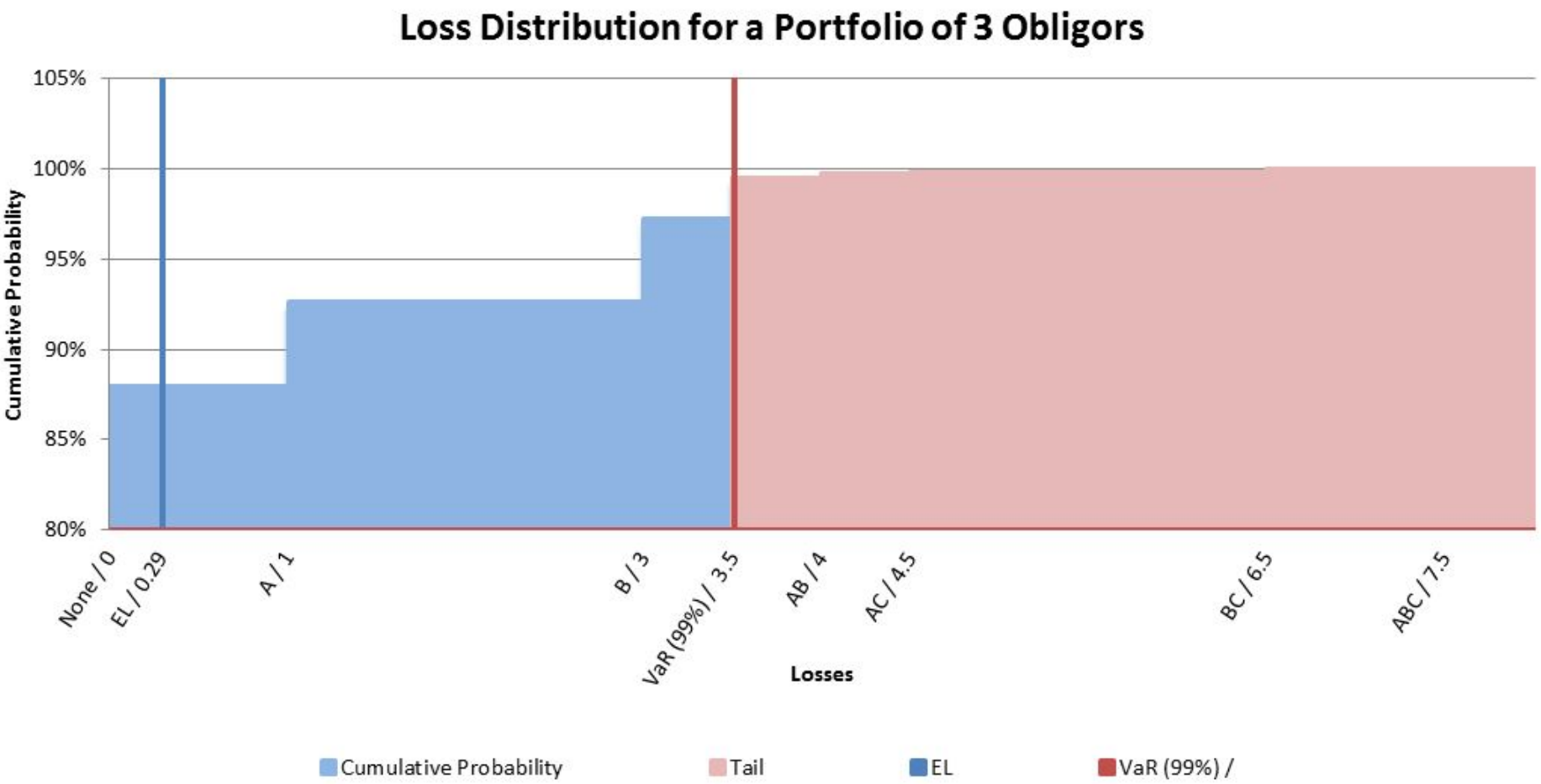

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